Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 2, pp. 207–212, 2009

GALILEAN-INVARIANT AXISYMMETRIC SELF-SIMILAR SUBMODEL OF GAS DYNAMICS WITHOUT SWIRLING

S. V. Khabirov

UDC 533

This paper deals with one insufficiently studied submodel of invariant solutions of rank 1 of the equations of gas dynamics. It is shown that, in cylindrical coordinates, the submodel without swirling reduces to a system of two ordinary differential equations. For the equation of state with additional invariance, a self-similar system is obtained. A pattern of phase trajectories is constructed, and particle motion is studied using asymptotic methods. The obtained solutions describe unsteady flows over axisymmetric bodies with possible strong discontinuities.

Key words: *gas dynamics, invariant submodel, strong discontinuities.*

Introduction. A program for obtaining and studying solutions of the equations of ideal gas dynamics on the basis of the admitted group of transformations is proposed in [1]. For the equation of state of the general form, such a group is the Galilean group extended by dilation. The basis for the classification of symmetric solutions is the optimum system of subalgebras of the Lie algebra of the admitted group. For each subalgebra of the optimum system, it is possible to construct a submodel using invariant, partially invariant or differential invariant solutions [2]. Invariant submodels are mostly listed in [3], and many of them have been investigated, for example, steady-state flows, one-dimensional motions, conical flows, etc. Invariant submodels of rank 1 reduce to ordinary differential equations and have been mostly investigated [4]. However, there are two three-dimensional subalgebras for which invariant submodels have been studied insufficiently, probably because of the presence of variables with fractional exponents in the coefficients of the quasilinear system of ordinary differential equations [4, § 18]. The present paper studies a particular case of one of the submodels for gas motion without swirling. Various unsteady flows over axisymmetric bodies in the longitudinal direction with the possible occurrence of shock waves or combustion waves are obtained. It is shown that some gas flows contain limiting surfaces where the acceleration is infinite. The method of research was developed on the basis of self-similar solutions of one-dimensional motions of an ideal polytropic gas [5].

1. Galilean-Invariant Axisymmetric Self-Similar Submodel. We consider subalgebra 3.3 of the optimum system of 11-dimensional algebra admitted by the equations of gas dynamics with an arbitrary equation of state [1]. The operator basis of the subalgebra consists of the Galilean translation operator $t\,\partial_x + \partial_U$, the rotation operator ∂_θ, and the dilation operator $t\,\partial_t + x\,\partial_x + r\,\partial_r$ in the cylindrical coordinates t, x, θ, and r. The invariants of these operators define the representation of the solution

$$U = x/t + U_1(s), \quad V = V(s), \quad W = W(s), \quad \rho = \rho(s), \quad S = S(s), \quad s = r/t, \tag{1.1}$$

where ρ is the density, S is the entropy, U, V, and W are the coordinates of the velocity; the pressure is defined by the equation of state $p = f(\rho, S)$.

In the cylindrical coordinates, the equations of gas dynamics are written as

$$U_t + UU_x + VU_r + r^{-1}WU_\theta + \rho^{-1}p_x = 0,$$

Institute of Mechanics, Ufa Scientific Center, Russian Academy of Sciences, Ufa 450054; habirov@anrb.ru.
Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 50, No. 2, pp. 46–52, March–April, 2009. Original article submitted June 4, 2008; revision submitted September 11, 2008.

$$V_t + UV_x + VV_r + r^{-1}WV_\theta + \rho^{-1}p_r = r^{-1}W^2,$$
$$W_t + UW_x + VW_r + r^{-1}WW_\theta + \rho^{-1}r^{-1}p_\theta = -r^{-1}VW, \tag{1.2}$$
$$\rho_t + U\rho_x + V\rho_r + r^{-1}W\rho_\theta + \rho(U_x + V_r + r^{-1}V + r^{-1}W_\theta) = 0,$$
$$S_t + US_x + VS_r + r^{-1}WS_\theta = 0.$$

Substitution of representation (1.1) into (1.2) yields the system of ordinary differential equations

$$(V - s)U_1' + U_1 = 0, \qquad \rho(V - s)V' + p' = 0, \qquad (V - s)sW' + VW = 0,$$
$$(V - s)s\rho' + \rho(sV' + V + s) = 0, \qquad (V - s)S' = 0. \tag{1.3}$$

Since for $\rho \neq 0$, we have $V \neq s$, the last equation in (1.3) implies the integral of entropy $S = S_0$, which is a constant. Then, $p' = f_\rho \rho' = a^2\rho'$ (a is the sound velocity).

There are two more integrals

$$W^3 s^2 = C_1\rho(V - s), \qquad U_1 = \mathcal{D}sW,$$

where C_1 and $\mathcal{D}$ are constants, which is easy to see after differentiation.

After the elimination of the function W, the submodel reduces to the system of two ordinary differential equations

$$V' + (V - s)\rho^{-1}\rho' = -1 - s^{-1}V,$$
$$(V - s)V' + a^2\rho^{-1}\rho' = C_1^{2/3}s^{-7/3}\rho^{2/3}(V - s)^{2/3}. \tag{1.4}$$

If the determinant of the system is equal to zero: $f_\rho = a^2 = (V - s)^2$, we obtain the special solution of the quasilinear systems (1.4) studied in [4, § 18]. In this case, the compatibility condition leads to the equality $C_1^2\rho^2 = s^4(V + s)^3(s - V)$, and relations (1.4) lead to the integral $s^5(s + V)^3(2s - 3V)^2 = -E$ $[(1 + s^{-1}V)E > 0$, where $E = \text{const}]$. This solution is possible only for the equation of state given parametrically:

$$C_1^2\rho^2 = E^{4/5}k(2 - k)^{3/5}(3k - 1)^{-8/5}, \qquad f_\rho = E^{1/5}k^2(2 - k)^{-3/5}(3k - 1)^{-2/5}$$

(k is a parameter). If $f_\rho \neq (V - s)^2$, system (1.4) can be solved with respect to the derivatives.

Let us consider the submodel without swirling ($C_1 = 0$). In this case, $W = U_1 = 0$. We obtain a nonautonomous quasilinear system but without fractional exponents in the coefficients. The first equation of system (1.4) is invariant under dilations $s \to \alpha s$, $V \to \alpha V$, and $\rho \to \beta\rho$ for any α and β. As a result of the above transformation, the second equation of system (1.4) becomes

$$(V - s)V' + \alpha^{-2}a^2(\beta\rho)\rho^{-1}\rho' = 0.$$

For this equation to be invariant, the function $a(\rho)$ should satisfy the functional equation $a(\beta\rho) = \alpha a(\rho)$. For $\alpha = \beta^{(\gamma-1)/2}$, the general solution of the functional equation becomes $a = \mathcal{D}\rho^{(\gamma-1)/2}$. From this it follows that the equation of state should have the form $p = \mathcal{D}^2(S)\gamma^{-1}(\rho^\gamma - 1) + S$. The additional symmetry of system (1.4) yields the independent system if this system is written in terms of the invariants $\rho = s^{2/(\gamma-1)}\rho_1$ and $V = sV_1$. Introducing the new function $\tau = \mathcal{D}^2\rho_1^{\gamma-1}$, we obtain a system which reduces to the independent system

$$((V_1 - 1)^2 - \tau)sV_1' = -V_1(V_1 - 1)^2 + (1 + 2V_1)\tau,$$
$$((V_1 - 1)^2 - \tau)\Big(s\tau^{-1}\tau' + 2\Big) = (\gamma - 1)(1 - V_1^2), \tag{1.5}$$

which is equivalent to the equation

$$\frac{d\tau}{dV_1} = \tau\,\frac{(1 - V_1)((\gamma + 1)V_1 + \gamma - 3) + 2\tau}{(1 + 2V_1)\tau - V_1(V_1 - 1)^2} \tag{1.6}$$

with rational right side.

The straight line $\tau = 0$ is an integral one. The integral curves of Eq. (1.6) have a physical meaning in the half-plane $\tau \geq 0$.

2. Integral curves. The parabola π of the extrema of the integral curves $\tau = \tau(V_1)$ is written as

$$\tau = (V_1 - 1)((\gamma + 1)V_1 + \gamma - 3)/2.$$

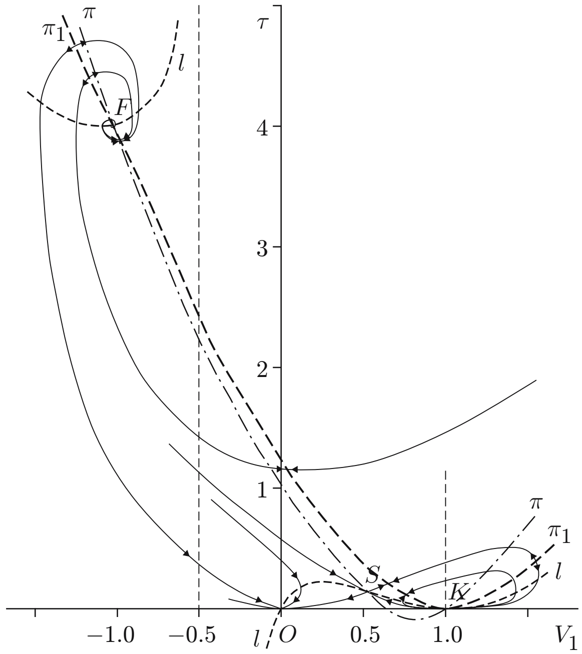

Fig. 1. Integral curve of Eq. (1.6) in the vicinity of the singular points: the arrows are the directions of increase of the quantity s.

The curve l of the extrema of the integral curves $V_1 = V_1(\tau)$ is written as

$$\tau = V_1(V_1 - 1)^2(2V_1 + 1)^{-1}.$$

This curve is doubly-connected, has the asymptote $V_1 = -1/2$, and touches the axis V_1 at the nonelementary singular point $K = (1, 0)$. The points of intersection of the curves of l and π and the straight line $\tau = 0$ are singular. Equation (1.6) has four singular points.

At the node $O = (0, 0)$, the behavior of the integral curves is represented as

$$V_1 \simeq \tau/(\gamma - 2) + C\tau^{1/(3-\gamma)} \tag{2.1}$$

(C is a constant). Under the assumption that $1 < \gamma < 2$, the integral curves touch the straight line $\tau = 0$; the exception is the distinguished branch for $C = 0$.

In the vicinity of the nonelementary singular point K, just as at the node, the integral curves touch the axis V_1 but doe not have the distinguished branch and are closed. In the vicinity of the point K, the asymptotic form of the integral curves is given by

$$\begin{aligned} \tau &\sim v^2\Big(\frac{2-\gamma}{3} - \frac{(2-\gamma)(\gamma+1)}{9(5-3\gamma)}\,v + C|v|^{2(2-\gamma)/(\gamma-1)}\Big) \quad \text{at} \quad \gamma \neq \frac{5}{3}, \\ \tau &\sim \frac{1}{9}\,v^2\Big(1 - \frac{4}{3}\,v\ln|v| + Cv\Big) \quad \text{at} \quad \gamma = \frac{5}{3}, \end{aligned} \tag{2.2}$$

where $v = V_1 - 1$ is a small quantity.

The point $S = (2^{-1}(3\gamma^{-1} - 1), (3/8)(3 - \gamma)(1 - \gamma^{-1})^2)$ is a saddle with tangents to the separatrices

$$8\tau - 3(3-\gamma)(1-\gamma^{-1})^2 = \Big[5\gamma - 9 \pm \sqrt{(5\gamma-9)^2 + 8(9-\gamma^2)(\gamma-1)}\Big]\Big(V_1 - (1/2)(3\gamma^{-1} - 1)\Big). \tag{2.3}$$

At the focus $F = (-1, 4)$, the behavior of the integral curves is given parametrically:

$$V_1 + 1 \sim -(1/4)C\exp(-\varphi\gamma_1)\cos\varphi, \qquad \tau - 4 \sim (\cos\varphi + \gamma_1^{-1}\sin\varphi)C\exp(-\varphi\gamma_1).$$

Here $\gamma_1 = \sqrt{2(1+\gamma)^{-1}}$ and φ is a parameter.

Figure 1 shows the integral curves. The parabola π_1 $[\tau = (V_1 - 1)^2]$ distinguishes a point on the integral curve that corresponds to the limiting surface in the region of motion of the gas on which the acceleration is infinite. This surface is sonic in the radial direction.

3. Gas Motions Corresponding to the Integral Curves in the Vicinity of Singular Points. The focus F corresponds to solution (1.1) of the form

$$U = xt^{-1}, \qquad V = -s, \qquad W = 0, \qquad \rho^{\gamma-1} = 4D^{-2}|s|^2,$$

$$p = D^2\gamma^{-1}\rho^{\gamma} + S, \qquad S = S_0, \qquad a = 2|s|, \qquad s = rt^{-1}.$$

Along the world line $\theta = \theta_0$, $r = r_0|t|^{-1}$, $x = x_0 t$, the gas-dynamic functions vary as follows:

$$\rho^{\gamma-1} = 4D^{-2}r_0^2 t^{-4}, \qquad a = 2r_0|t|^{-2}, \qquad V = -r_0 t^{-2}, \qquad U = x_0.$$

The trajectories are hyperbolas whose asymptotes are the r and x axes. At $t > 0$, particles from a circle at infinity ($r \to \infty$ as $t \to 0$) focus to the axis x: $r \to 0$ and $x \to \infty$ ($U \to x_0$, $V \to 0$, and $\rho \to 0$ as $t \to \infty$). This behavior corresponds to the formation of a jet. At $t < 0$ and as $t \to -\infty$, particles from infinity on the x axis tend to a circle at infinity: $r \to \infty$, $x \to 0$ as $t \to -0$ ($U \to x_0$, $V \to \infty$, and $\rho \to \infty$) which corresponds to jet impingement on a wall.

The saddle point S corresponds to the solution

$$U = \frac{x}{t}, \qquad V = \Big(\frac{3}{2\gamma} - \frac{1}{2}\Big)\frac{r}{t}, \qquad W = 0, \qquad a^2 = \frac{3}{8}(3-\gamma)(1-\gamma^{-1})^2\,\frac{r^2}{t^2}. \tag{3.1}$$

For $\gamma = (2 + \sqrt{13}\,)/3$, we have sonic motion: $|V| = a$.

At the point $x = 0$, $r = 0$ as $t \to -0$, the world lines of particles

$$\theta = \theta_0, \qquad x = x_0 t, \qquad r = r_0|t|^{(3-\gamma)/(2\gamma)}$$

collapse, which is followed by an instantaneous explosion.

At the node O for $\gamma = 3/2$, the asymptotic (2.1) has the form

$$V_1 \sim -2\tau + C\tau^{2/3},$$

where $C \neq 0$ is a parameter of the integral curves and τ is a small parameter. System (1.5) defines the asymptotics of the quantity $s \sim \mathcal{D}(-(2/3)C + \tau^{-2/3})$ ($\mathcal{D}$ is a constant parameter of various solutions).

The equation of the world lines of particles $dr/dt = V$ defines the quantities t and $r = ts$ depending on τ for $C \neq 0$:

$$t \sim N\tau^{2/3}(1 + (5/3)C\tau^{2/3}), \qquad r \sim NC(1 + C\tau^{2/3}) \sim C(N + (3/10)Ct)$$

(N is a constant that characterizes a particle).

As $\tau \to 0$, $r \to NC$, $t \to 0$, and $V \to (3/10)C^2$, and the particles begin to move at identical velocities.

For $C = 0$, the asymptotics have the form

$$V_1 \sim -2\tau, \qquad s \sim \mathcal{D}\tau^{-2/3}(1 + (2/3)\tau), \qquad t \sim N(\tau^{2/3} - 2\tau^{5/3}),$$

$$r \sim \mathcal{D}N(1 - (4/3)\tau), \qquad V \sim -2\mathcal{D}\tau^{1/3}(1 + (2/3)\tau).$$

For $t \to -0$, $\mathcal{D} < 0$, $N < 0$, the motion ends with ($V \to 0$) infinite deceleration. For $t \to 0$, $\mathcal{D} > 0$, $N > 0$, the motion begins at zero velocity and infinite acceleration. Since, for a particle, $x = x_0 t$, the trajectory from the coordinate origin can be considered a wall. This motion can be treated as unsteady flow over a wall.

In the vicinity of the point $K = (1, 0)$ for $\gamma = 3/2$, the asymptotic (2.2) becomes

$$\tau \sim (1/6)v^2 - (5/18)v^3 + Cv^4, \qquad v = V_1 - 1.$$

From relations (1.5), we obtain

$$s \sim \mathcal{D}(1 - (5/3)v).$$

The equations for the world lines imply that

$$t \sim N|v|^{-5/3}, \qquad r \sim N\mathcal{D}|v|^{-5/3}(1 - (5/3)v).$$

As $v \to 0$, $t \to \infty$, $r \to \infty$, $V \to \mathcal{D}$, and the expansion of particles to infinity occurs at the same finite velocity.

The integral curve connecting the singular points O and K, describes the expansion to infinity at identical velocity for the particles flowing around an axisymmetric body.

For $\gamma = 3/2$, the asymptotics (2.3) of the separatrices of the saddle S generate the solution $s = C|v|^{(2\pm\sqrt{13})/6}$ and the particle motion:

$$t \sim N|v|^{-(2\pm\sqrt{13})/3}, \qquad r \sim NC|v|^{-(2\pm\sqrt{13})/6} \sim C|\mathcal{D}t|^{1/2}.$$

Similarly to the steady-state solution (3.1) for the separatrix with the plus sign, we obtain a collapse and an instantaneous source. For the separatrix with the minus sign at $v \to 0$, we have $t \to \infty$ and $r \to \infty$, which corresponds to the expansion of the gas at infinite velocity at infinity.

At the points of the parabola π_1, the asymptotic form of the solutions of system (1.5) is given by

$$\tau_0 = (V_{10} - 1)^2.$$

Eliminating the singular points, we obtain

$$\tau - (V_{10} - 1)^2 \sim -\frac{1}{2}(V_{10} - 1)v + \frac{9(2 - 3V_{10})}{8(V_{10} + 1)}v^2, \qquad s \sim C\Big(1 + \frac{5v^2}{4(1 - V_{10}^2)}\Big),$$

where $v = V_1 - V_{10}$ is a small parameter and C and V_{10} are constants of the solutions. The asymptotic form of the world lines is given parametrically:

$$t \sim N\Big(1 + \frac{5v^2}{4(1 - V_{10}^2)(V_{10} - 1)}\Big), \qquad r = ts \sim CN\Big(1 + \frac{5V_{10}v^2}{4(1 - V_{10}^2)(V_{10} - 1)}\Big)$$

(N is a constant that defines the world line of the particle).

As $v \to 0$, the particles are on the straight line $r = Ct$ and have identical velocities CV_{10}. As $v \to +0$ and $v \to -0$, the particles move on different curves (the terms of the asymptotics with odd powers of opposite sign) but on the same side of the limiting straight line. Hence, in the same region bounded by the limiting sonic (in the radial direction) line, we have two different gas flows. In Fig. 1, this is shown by arrows whose directions on the different sides from the parabola π_1 are opposite. The region above the parabola π_1 corresponds to subsonic motion, and the region below the parabola π_1 to supersonic motion.

4. Strong Discontinuities. The motions on different sides pf the sonic parabola π_1 can be united in one motion using invariant shock or detonation transition at the points of the integral curves adjacent to this parabola.

We consider a polytropic gas for which the equation of state is given by

$$p = \gamma^{-1}\mathcal{D}(S)^2\rho^\gamma.$$

The invariant surface of a strong discontinuity is given by the equality $r = Et$ (E is the constant velocity). The condition on the discontinuity can be written in invariant form [5, 4,§ 2]:

$$[W] = [U] = 0,$$

$$\rho_1(V_{11} - 1) = \rho_2(V_{12} - 1),$$

$$p_1 + \rho_1(V_{11} - 1)^2E^2 = p_2 + \rho_2(V_{12} - 1)^2E^2, \tag{4.1}$$

$$\frac{\gamma_1 p_1}{(\gamma_1 - 1)\rho_1} + \frac{1}{2}(V_{11} - 1)^2E^2 + Q = \frac{\gamma_2 p_2}{(\gamma_2 - 1)\rho_2} + \frac{1}{2}(V_{12} - 1)^2E^2.$$

Here Q is the energy influx to unit mass in a detonation wave or in a combustion wave. In a shock wave, $Q = 0$ and $\gamma_1 = \gamma_2 = \gamma$.

Using the variables τ and V_1, we represent the discontinuity conditions (4.1) as

$$\frac{\tau_2}{\gamma_2(V_{12} - 1)} + V_{12} = \frac{\tau_1}{\gamma_1(V_{11} - 1)} + V_{11},$$

$$\frac{2\tau_2}{\gamma_2 - 1} + (V_{12} - 1)^2 = Q_1 + \frac{2\tau_1}{\gamma_1 - 1} + (V_{11} - 1)^2, \tag{4.2}$$

where $Q_1 = 2QE^{-2} > 0$. In the combustion wave, rarefaction $\tau_2 < \tau_1$ occurs in the subsonic region $\tau_i > (V_{1i} - 1)^2$. The detonation wave is a compression wave $\tau_2 > \tau_1$ with transition from the supersonic region to the subsonic region or the sonic line (the Chapman–Jouguet condition).

The patterns of the integral curves for various values of γ differ only in the location of the saddle below the limiting sonic parabola in the supersonic region.

Relations (4.2) transform a point (V_{11}, τ_1) ahead of the front to a point (V_{12}, τ_2) behind the front. Thus, for a detonation wave, the curve from the supersonic region is transformed to the sonic parabola π_1:

$$\frac{\tau_1}{(V_{11}-1)^2} = \frac{\gamma_1}{\gamma_2^2(\gamma_1-1)}\Big[\gamma_2^2 - \gamma_1 + \sqrt{\gamma_2^2-1}\sqrt{\gamma_2^2-\gamma_1^2+Q_1\gamma_2^2(\gamma_1-1)^2(V_{11}-1)^{-2}}\,\Big].$$

For the shock wave, the sonic parabola remains invariant.

If a point behind the front is in the subsonic region, the further motion along the integral curve toward the arrows leads to the limiting sonic line. Thus, a strong discontinuity cannot eliminate the limiting line in gas motion, but, on the contrary, it leads to its occurrence.

This work was supported by the Russian Foundation for Basic Research (Grant No. 08-01-00047-a) and Council for Grants of the President of the Russian Federation for Support of Leading Scientific Schools (Grant No. NSh-2826.2008.1).

REFERENCES

1. L. V. Ovsyannikov, "The program SUBMODELS. Gas dynamics," *Prikl. Mat. Mekh.*, **58**, No. 4, 30–55 (1994).
2. L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York (1982).
3. S. V. Khabirov, "Invariant solutions of the equations of gas dynamics," *Vestnik Ufa Gos. Avaits.-Tekh. Univ.*, No. 1, 47–52 (2001).
4. S. V. Khabirov, *Analytical Methods in Gas Dynamics* [in Russian], Gilem, Ufa (2003).
5. L. I. Sedov, *Methods of Similarity and Dimension in Mechanics* [in Russian], Nauka, Moscow (1965).